CAUCHY TYPE DOUBLE INTEGRAL REPRESENTATION FOR FUNCTIONS OF A COMPLEX VARIABLE

BY

CHARLES WILLARD MATHEWS, JR.
B.Ed., Southern Illinois Normal University, 1937
M.A., University of Illinois, 1939

AN ABSTRACT OF A THESIS

SUBMITTED IN PARTIAL FULFILLMENT OF THE REQUIREMENTS
FOR THE DEGREE OF DOCTOR OF PHILOSOPHY IN MATHEMATICS
IN THE GRADUATE SCHOOL OF THE UNIVERSITY
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Consider functions $f(z) = u(x,y) + i\, v(x,y)$ of the complex variable $z = x + i\, y$ which are continuous in G, a bounded domain in the complex plane, whose boundary, $\bar{G}-G$, has zero two-dimensional measure. Assume $f(z)$ such that

$$\overline{\lim}\ \frac{1}{|I|} \int_{(I)} f(z)dz < +\infty \qquad \text{(intervals, } I \supset z_0\text{)}$$

for all z_0 in $G-S$, where S is a denumerable infinity (at most) of segments parallel to the axes; moreover, the partial derivatives, u_x, u_y, v_x, and v_y, are assumed to exist almost everywhere in G, say in $G-G_0$, where the measure of G_0 is zero. Functions which satisfy these conditions will be called functions of class R.

W. J. TRJITZINSKY* has found conditions under which $f(z)$, belonging to R, can be represented as a Cauchy type integral,

$$(1) \qquad f(z) = -\frac{1}{2\pi i} \iint_G \frac{J'(\zeta)d\xi_1 d\xi_2}{\zeta - z} + a(z)$$

where $a(z)$ is analytic in G, and $J'(z_0)$ is the ordinary interval derivative at z_0 of $J(I) = \int_{(I)} f(z)dz$.

In order to show conditions under which equations of type (1) are equivalent to regular Fredholm integral equations it is first necessary to establish an auxillary theorem. Consider the equation:

$$(2) \qquad \iint_G \frac{\Phi(\zeta)d\xi_1 d\xi_2}{\zeta - z} + \iint_G K(z,\zeta)\Phi(\zeta)d\xi_1 d\xi_2 - f(z) = a(z)$$

where $a(z)$ is analytic in G but not preassigned and $\Phi(\zeta)$ is the unknown function. Associate with (2) the equation:

$$(3) \qquad 2\pi i \Phi(z) - \iint_G \Phi'(z,\zeta)\Phi(\zeta)d\xi_1 d\xi_2 = -\eta'(z)$$

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* W. J. TRJITZINSKY, Theory of Functions of an Interval and Applications to Functions of a Complex Variable, to appear in Journal de Mathematiques, see Theorem 6'.6.

where $\Phi(I,\eta) = \int_{(I)} K(z,\eta)\,dz$ (for $I \subset G$)

$\eta(I) = \int_{(I)} f(z)\,dz$ (for $I \subset G$)

LEMMA 4: EVERY SOLUTION $\phi(\eta)$ OF (2), BOUNDED ALMOST EVERYWHERE IN G, WILL BE A SOLUTION OF THE REGULAR FREDHOLM EQUATION (3), AND CONVERSELY, PROVIDED $f(z)$ AND $K(z,\eta)$ SATISFY THE FOLLOWING CONDITIONS:

(4.1) $f(z) \subset R$,

(4.2) $\eta'(\eta) \subset L_2$,

(4.3) $g(z) = \iint_G K(z,\eta)\,\phi(\eta)\,d\eta_1\,d\eta_2 \subset R$ FOR ALL $\phi(\eta) \subset L_2$,

(4.4) $$\int_{(I)} \left[\iint_G K(z,\eta)\,\phi(\eta)\,d\eta_1 d\eta_2\right] dz = \iint_G \left[\int_{(I)} K(z,\eta)\,dz\right]\phi(\eta)\,d\eta_1 d\eta_2$$

FOR ALL $\phi(\eta) \subset L_2$, ALL $I \subset G$,

(4.5) $D_z\Phi(I,\eta) = \Phi'(z,\eta) \subset L_2$ IN z AND η, AND

(4.6) $D_z \iint_G \Phi(I,\eta)\,\phi(\eta)\,d\eta_1 d\eta_2 = \iint_G D_z\Phi(I,\eta)\,\phi(\eta)\,d\eta_1 d\eta_2$

FOR $\phi(\eta) \subset L_2$.

$D_z\ldots$ denotes the ordinary derivative of the interval function at z. Under these conditions TRJITZINSKY* has shown that if $\mu(e)$ is defined as

$$\mu(e) = \iint_e \phi(\eta)\,d\eta_1 d\eta_2 \qquad \text{(Borel set, } e \subset G\text{)}$$

that

$$\mu(I) = \iint_I \phi(\eta)\,d\eta_1 d\eta_2 = -\frac{1}{2\pi i}\int_{(I)} [f(z)-g(z)]\,dz.$$

From this point the proof that (2) is equivalent to (3) is direct and reversible.

THEOREM 5: THE INTEGRAL EQUATION (2) IS EQUIVALENT TO THE INTEGRAL EQUATION (3) IN THE SENSE THAT A SOLUTION, $\phi(\eta)$,

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* W. J. TRJITZINSKY, Problems of Representation and Uniqueness for Functions of a Complex Variable, Acta Mathematica, Vol. 78 (1946) p. 135.

BELONGING TO L_2 IN G, OF ONE IS A SOLUTION OF THE OTHER, PROVIDED f(z) BELONGS TO R AND

(5.1) $\iint_G |K^2(\alpha, \eta)|\, d\eta_1 d\eta_2 < +\infty$ (FOR SOME α IN G)

$\iint_G |K^2(z, \eta) - K(z', \eta)|\, d\eta_1 d\eta_2$ EXISTS FOR ALL z AND z' AND APPROACHES ZERO WITH $|z - z'|$,

(5.2) $\int_\alpha^x \left[\iint_G \left|\frac{\partial K(x, y, \eta_1, \eta_2)}{\partial x} \Phi(\eta_1, \eta_2)\right| d\eta_1 d\eta_2\right] dx$ EXISTS FOR ALMOST ALL y AND

$\int_\beta^y \left[\iint_G \left|\frac{\partial K(x, y, \eta_1, \eta_2)}{\partial y} \Phi(\eta_1, \eta_2)\right| d\eta_1 d\eta_2\right] dy$ EXISTS FOR ALMOST ALL x,

(5.3) $\frac{|\Phi(I, \eta)|}{|I|} \leqq h(z) q(\eta)$ (FOR ALL $I \supset z$; $h(z) \neq \infty$ in G-S; $q^2(\eta)$ INTEGRABLE OVER G),

(5.4) $\frac{|\Phi(I, \eta) - \Phi(I, \eta')|}{|I|} \leqq h(z) q(\eta, \eta')$ (FOR ALL $I \supset z$, $q(\eta, \eta') \to 0$ AS $|\eta - \eta'| \to 0$), AND

(5.5) $\Phi'(z, \eta) \subset L_2$ IN z AND η.

The proof is obtained by showing that the conditions of LEMMA 4 are satisfied under the assumptions of the theorem.

Thus, it is seen that if f(z) belongs to R and $K(z, \eta)$ satisfies the conditions of THEOREM 5, while the equation (3) has a solution, $\Phi(\eta) \subset L_2$, then f(z) has the representation

$$f(z) = -\iint_G \frac{k(z, \eta)\Phi(\eta)\, d\eta_1 d\eta_2}{\eta - z} + a(z)$$

where

$$K(z, \eta) = \frac{k(z, \eta) - 1}{\eta - z}$$

and a(z) is analytic in G.

VITA

Charles Willard Mathews, Jr., was born in St. Louis, Missouri, on September 11, 1915. He attended public schools in Marissa, Illinois, and graduated from the Marissa Township High School in 1933. From 1933 to 1937 he attended Southern Illinois Normal University at Carbondale, Illinois, and received the Bachelor of Education degree in June, 1937. He was principal of the Fisk, Missouri, High School for the school year, 1937-1938. He received the Master of Arts degree from the University of Illinois in June, 1939. From September, 1939, to June, 1942, and from March, 1946, to June, 1947, he was a graduate assistant at the University of Illinois. In the interim he served as an officer in the United States Navy, instructing in ordnace and fire control at the United States Naval Reserve Midshipmen School, Columbia University, and serving as operations officer at the port of Naha, Okinawa.

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